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Marginal Distribution of Wigner Function in Mesoscopic RLC Circuit at Finite Temperature and Its Application

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Abstract By means of the Weyl correspondence and Wigner theorem the marginal distribution of Wigner function in mesoscopic RLC circuit at finite temperature was discussed. Here we endow the Wigner function with a new physical meaning, i.e., its marginal distributions' statistical average for $q^2/(2C)$ and $p^2/(2L)$ are the temperature-related energy stored in capacity and in inductance of the mesoscopic RLC circuit, respectively.

Keywords Mesoscopic RLC circuit · IWOP technique · Wigner function · Marginal distribution

1 Introduction

With the development of the solid-state quantum computation, it becomes more and more important to study the quantum characteristic of mesoscopic systems. Louisell is the first physicist who proposed quantization scheme for LC circuit [1] and some more progress in this field has been made in recent years [2–4]. Wang et al. [5] investigated the quantization of mesoscopic RLC circuit, and the quantum fluctuation in the situation at zero temperature was discussed. For such a circuit including dissipation, Joule heat will generate when electric current flows along the resistance, and thermo noise will affect the stability and precision of the signal. So it is constructive to consider RLC mesoscopic circuit at finite temperature. In this paper, based on [5], we will treat the RLC mesoscopic circuit by taking into account the temperature effect.

Generally, one of the theories for dealing with quantum system at finite temperature is to use the thermo field dynamics (TFD) [6, 7], which was established by Takahashi and Umezawa for converting the calculations of ensemble averages at nonzero temperature T into equivalent expectation values with a pure state, this worthwhile convenience is at the expense of introducing a fictitious field (or called a tilde-conjugate field) in the extending

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Hilbert space, thus the original optical field state $|n\rangle$ in the Hilbert space is accompanied by a tilde state $|\widetilde{n}\rangle$ in fictitious space. For example, let $\rho(\hat{a},\hat{a}^\dagger)$ be a density matrix of the system under review, the thermal ensemble average of an operator in the original Fock space can then be written as [8, 9]

$$\langle 0(\beta)|A|0(\beta)\rangle = \mathrm{Tr}(Ae^{-\beta\hat{\mathcal{H}}})/\mathrm{Tr}(e^{-\beta\hat{\mathcal{H}}}), \tag{1}$$

where the thermal vacuum $|0(\beta)\rangle$, ($\beta = \frac{1}{kT}$, k is the Boltzmann constant), is defined by Takahashi and Umezawa such that the vacuum expectation value agrees with the statistical average, $\hat{\mathcal{H}}$ is the system's Hamiltonian operator. Takahashi and Umezawa also introduced the thermo operator $\hat{S}(\theta)$, which engenders the transition from zero temperature to finite temperature,

$$\hat{S}(\theta)|0,\tilde{0}\rangle = |0(\beta)\rangle, \tag{2}$$

where the vacuum state $|0,\tilde{0}\rangle$ is annihilated by either $\hat{a}$ or $\widetilde{\hat{a}}$, and

$$\begin{aligned}\hat{S}(\theta) &= \exp[\theta(\hat{a}^\dagger\widetilde{\hat{a}}^\dagger - \hat{a}\widetilde{\hat{a}})]\\ &= \exp(\hat{a}^\dagger\widetilde{\hat{a}}^\dagger\tanh\theta)\exp[(\hat{a}^\dagger\hat{a} + \widetilde{\hat{a}}^\dagger\widetilde{\hat{a}} + 1)\ln\sec h\theta]\exp(-\hat{a}\widetilde{\hat{a}}\tanh\theta).\end{aligned} \tag{3}$$

2 Quantization Scheme for RLC Circuit with a Source

Let us briefly review the quantization scheme for RLC circuit with a source $\varepsilon(t)$ in [5]. The classical Kirchhoff equation for the RLC circuit is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = \varepsilon(t) \tag{4}$$

where L, R and C stand for inductance, resistance and capacity, respectively, q is the charge. From (4), we see that energy stored in the circuit includes: (1) increase of the circuit energy $E_1 = -q\varepsilon(t)$, (2) capacity energy $E_2 = q^2/2C$, (3) inductance energy $E_3 = (1/2)L(dq/dt)^2$, (4) energy consumed by the resistance $E_4 = R\frac{dq}{dt}q$.

Therefore, the total energy change of the system is

$$E = \frac{p^2}{2L} + \frac{R}{L}qp + \frac{q^2}{2C} - q\varepsilon(t). \tag{5}$$

From the point of view of classical Hamiltonian dynamics, we denote the electric charge q generalized "coordinate" and $p(t) = L(dq/dt)$,which in fact is the magnetic flux through the inductance, the generalized momentum. By introducing the quantization condition $[\hat{q},\hat{p}] = i\hbar$, the quantized Hamiltonian of the system is

$$\hat{H} = \frac{\hat{p}^2}{2L} + \frac{R}{2L}(\hat{q}\hat{p} + \hat{p}\hat{q}) + \frac{\hat{q}^2}{2C} - \hat{q}\varepsilon(t). \tag{6}$$

When the source is considered as a pulse signal, if the exciting time $t \to 0$, the Hamiltonian of the circuit system can be written as

$$\hat{H}_0 = \frac{\hat{p}^2}{2L} + \frac{R}{2L}(\hat{q}\hat{p} + \hat{p}\hat{q}) + \frac{\hat{q}^2}{2C}. \tag{7}$$

Introducing the unitary operator [5]

$$\hat{U} = \exp\left(i\frac{R}{2\hbar}\hat{q}^2\right), \tag{8}$$

to make transformation for (7) we have

$$\widetilde{\hat{H}_0} = \hat{U}\hat{H}_0\hat{U}^{-1} = \frac{\hat{p}^2}{2L} + \frac{1}{2}L\omega'^2\hat{q}^2 = \hbar\omega'\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{9}$$

where

$$\hat{a} = \sqrt{2L\hbar\omega'}(\omega' L\hat{q} + i\hat{p}), \qquad \hat{a}^\dagger = \sqrt{2L\hbar\omega'}(\omega' L\hat{q} - i\hat{p}), \tag{10}$$

respectively, represent bosonic creation operator and bosonic annihilation operator, $\omega' = \omega_0\sqrt{1 - R^2C/L}$, $(\omega_0 = 1/\sqrt{LC})$ is the resonant frequency of RLC circuit. From (10), we have

$$\hat{q} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}\sqrt{\frac{\hbar}{\omega' L}}, \qquad \hat{p} = \frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2}i}\sqrt{L\hbar\omega'}. \tag{11}$$

In the framework of the TFD the operator $\hat{a}^\dagger$ is now accompanied with $\widetilde{\hat{a}}^\dagger$, for $\widetilde{\hat{H}_0}$ the corresponding thermal vacuum state $|0(\beta)\rangle$ is

$$|0(\beta)\rangle = \sec h\theta \exp\left[\hat{a}^\dagger\widetilde{\hat{a}}^\dagger \tanh\theta\right]|0, \tilde{0}\rangle, \quad \tanh\theta = \exp\left(-\frac{\hbar\omega'}{2kT}\right). \tag{12}$$

3 Wigner Function of Mesoscopic RLC Circuit and Its Marginal Distribution

We now tackle mesoscopic RLC circuit at finite temperature by means of the Weyl correspondence and Wigner theorem,

$$\begin{aligned}\langle\hat{A}\rangle &= \iint dq dp W_T(q,p)\langle\triangle_w(q,p)\rangle \\ &= 2\int W_T(\alpha,\alpha^*)\langle\triangle_w(\alpha,\alpha^*)\rangle d^2\alpha, \quad \alpha = \frac{q+ip}{\sqrt{2}},\end{aligned} \tag{13}$$

where $W_T(q,p)$ is the classical correspondence of operator $\hat{A}$, $\triangle_w(q,p) = \triangle_w(\alpha,\alpha^*)$ is the Wigner operator, the subscript "T" means "thermal". Using the coherent state's completeness relation $\int \frac{d^2z}{\pi}|z\rangle\langle z| = 1$ and the technique of integration within an ordered product (IWOP) [9] of operator, the Wigner function of the thermo vacuum state is

$$\begin{aligned}W_T(q,p) &= \langle 0(\beta)|\triangle_w(q,p)|0(\beta)\rangle \\ &= \frac{1 - e^{-\beta\hbar\omega'}}{\pi(1 + e^{-\beta\hbar\omega'})}\exp\left[-\frac{1 - e^{-\beta\hbar\omega'}}{1 + e^{-\beta\hbar\omega'}}(q^2 + p^2)\right].\end{aligned} \tag{14}$$

Thus, the marginal distribution of the Wigner function of the thermal vacuum state for RLC circuit in the coordinate space and momentum space are, respectively,

$$\begin{aligned}\int dq W_T(q,p) &= \frac{1}{\sqrt{\pi}}\sqrt{\frac{1 - e^{-\beta\hbar\omega'}}{1 + e^{-\beta\hbar\omega'}}}\exp\left[-\frac{1 - e^{-\beta\hbar\omega'}}{1 + e^{-\beta\hbar\omega'}}p^2\right] \\ &= \frac{1}{\sqrt{\pi}}\tanh^{1/2}(\beta\hbar\omega'/2)\exp[-p^2\tanh(\beta\hbar\omega'/2)],\end{aligned} \tag{15}$$

$$\int dp W_T(q,p) = \frac{1}{\sqrt{\pi}} \tanh^{1/2}(\beta\hbar\omega'/2) \exp[-q^2 \tanh(\beta\hbar\omega'/2)]. \tag{16}$$

Because the classical Weyl correspondence of $\hat{q}$, $\hat{p}$ is

$$\hat{q} \to \frac{\alpha+\alpha^*}{\sqrt{2}}\sqrt{\frac{\hbar}{\omega' L}}, \qquad \hat{p} \to \frac{\alpha-\alpha^*}{\sqrt{2}i}\sqrt{L\hbar\omega'}. \tag{17}$$

By virtue of the Weyl-Wigner rule and (16) and (17), we obtain

$$\begin{aligned}
\left\langle(\triangle\hat{q})^2\right\rangle &= \langle 0(\beta)|\hat{q}^2|0(\beta)\rangle = \int d^2\alpha \frac{\hbar(\alpha+\alpha^*)^2}{\omega' L}\langle 0(\beta)|\triangle_w(\alpha,\alpha^*)|0(\beta)\rangle \\
&= \frac{\hbar}{2\omega' L}\coth\frac{\beta\hbar\omega'}{2} \\
&= \frac{\hbar}{2L(\sqrt{LC-R^2C^2})}\coth\frac{\beta\hbar(\sqrt{LC-R^2C^2})}{2}
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
\left\langle(\triangle\hat{p})^2\right\rangle &= \frac{\hbar\omega' L}{2}\coth\frac{\beta\hbar\omega'}{2} \\
&= \frac{\hbar L(\sqrt{LC-R^2C^2})}{2}\coth\frac{\beta\hbar(\sqrt{LC-R^2C^2})}{2}
\end{aligned} \tag{19}$$

The uncertainty relation for the charge and the current is

$$\langle\triangle\hat{q}\rangle\langle\triangle\hat{p}\rangle = \frac{\hbar}{2}\coth\frac{\beta\hbar\omega'}{2} = \frac{\hbar}{2}\coth\frac{\beta\hbar(\sqrt{LC-R^2C^2})}{2}. \tag{20}$$

One can clearly see that since $\beta = 1/(kT)$, the fluctuation and uncertainty relation of $\hat{q}$, $\hat{p}$ are affected by the temperature.

Further, we can calculate the temperature-related energy stored in the capacitance

$$\begin{aligned}
E_C &= \langle 0(\beta)|\frac{\hat{q}^2}{2C}|0(\beta)\rangle = \frac{1}{2C}\int dpdq \frac{\hbar(\alpha+\alpha^*)^2}{2\omega' L}\langle 0(\beta)|\triangle_w(p,q)|0(\beta)\rangle \\
&= \frac{1}{2C}\int dq \frac{\hbar(\alpha+\alpha^*)^2}{2\omega' L}\int dp W_T(q,p) \\
&= \frac{\hbar\omega_0^2}{4\omega'}\coth\frac{\beta\hbar\omega'}{2} \\
&= \frac{\hbar}{4LC(\sqrt{LC-R^2C^2})}\coth\frac{\beta\hbar(\sqrt{LC-R^2C^2})}{2}
\end{aligned} \tag{21}$$

and the temperature-related energy stored in the inductance is

$$\begin{aligned}
E_L &= \langle 0(\beta)|\frac{\hat{p}^2}{2L}|0(\beta)\rangle = \frac{1}{2L}\int dpdq \frac{L\hbar\omega'(\alpha-\alpha^*)^2}{2}\langle 0(\beta)|\triangle_w(p,q)|0(\beta)\rangle \\
&= \frac{1}{2L}\int dp \frac{L\hbar\omega'(\alpha-\alpha^*)^2}{2}\int dq W_T(q,p)
\end{aligned}$$

$$= \frac{\hbar\omega'}{4}\coth\frac{\beta\hbar\omega'}{2}$$
$$= \frac{\hbar(\sqrt{LC - R^2C^2})}{4}\coth\frac{\beta\hbar(\sqrt{LC - R^2C^2})}{2}. \quad (22)$$

Here we endow the Wigner function with a new physical meaning, i.e., its marginal distributions' statistical average for $q^2/(2C)$ and $p^2/(2L)$ are the temperature-related energy stored in the capacitance and in the inductance of the mesoscopic RLC circuit at finite temperature, respectively.

4 Conclusion

In summary, we consider the marginal distribution of Wigner function in mesoscopic RLC circuit at finite temperature. By virtue of the thermo field dynamics we show that the quantum mechanical zero-point fluctuations of both charge and current increase with the rising of temperature and the resistance value, i.e., the RLC circuit at finite temperature exhibits more noise than at $T \to 0$ case. Marginal distributions' statistical average of the Wigner function for $q^2/(2C)$ and $p^2/(2L)$ are the temperature-related energy stored in the capacitance and in the inductance of the mesoscopic RLC circuit. This new view endows the Wigner function with a more realistic physical interpretation.

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